OSCILLATORY APPROACH TO THE SINGULARITY IN VACUUM T^2 SYMMETRIC SPACETIMES

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A combination of qualitative analysis and numerical study indicates that vacuum T^2 symmetric spacetimes are, generically, oscillatory.

Thirty years ago, Belinskii, Khalatnikov and Lifshitz proposed that the dynamics of spatially inhomogeneous solutions to Einstein's equation near a cosmological singularity are oscillatory in general.[1] The quantities which oscillate are the generalized Kasner exponents, which are the eigenvalues of the extrinsic curvature divided by the mean curvature. In various known cases, there exists a foliation and threading of a solution to Einstein's equation in the neighborhood of a cosmological singularity such that the generalized Kasner exponents along each thread either converge to a limit at the singularity (if this occurs on all threads we call the solution convergent), or approximately follow the BKL sequence.[1] Along threads with oscillatory behavior, the BKL sequence is realized because the evolution becomes approximately Kasner, but then there is inevitably a transition to a different Kasner evolution. Each transition can be approximated, yielding the BKL sequence. The singularity is at finite proper time, but there are an infinite number of oscillations.

Numerical simulations of vacuum T^2 symmetric spacetimes show that an "asymptotic regime" is reached in which the generalized Kasner exponents follow a portion of the BKL sequence. Qualitative analysis in the asymptotic regime indicates that the oscillations will, in general, continue without end at almost every spatial point. These methods have been described previously.[2,3,4]

Gowdy spacetimes are vacuum T^2 symmetric spacetimes such that the (spacelike) symmetry orbits are surface orthogonal. It is thought that all Gowdy spacetimes are convergent. We find oscillatory dynamics in non-polarized T^2 symmetric spacetimes if the symmetry orbits are not surface orthogonal, in which case the spatial topology must be the three torus. A global foliation and threading of these spacetimes is known.[5] The metric can be written

$$\begin{aligned} g = &-e^{\frac{\lambda-3\tau}{2}} d\tau^2 + e^{\frac{\lambda+\mu+\tau}{2}} d\theta^2 + \sigma\, e^{P-\tau}[dx + Q\, dy + (G_1 + Q\, G_2)\, d\theta \\ &- (M_1 + Q\, M_2)\, e^{-\tau} d\tau]^2 + \sigma\, e^{-P-\tau}[dy + G_2\, d\theta - M_2\, e^{-\tau} d\tau]^2, \end{aligned} \tag{1}$$

with the singularity in the direction of increasing τ. The metric functions are

independent of x and y and periodic in θ. If Q vanishes everywhere the solution is polarized, and thought to be convergent.[6] If the non-positive function $\partial_\tau \mu$ vanishes everywhere, the symmetry orbits are surface orthogonal. If $|\partial_\tau \mu| \ll 1$, then the generalized Kasner exponents are approximately

$$\frac{\partial_\tau \lambda + \partial_\tau \mu + 1}{\partial_\tau \lambda + \partial_\tau \mu - 3} \qquad \frac{2\,(v-1)}{\partial_\tau \lambda + \partial_\tau \mu - 3} \qquad \frac{-2\,(v+1)}{\partial_\tau \lambda + \partial_\tau \mu - 3}. \tag{2}$$

The error is less than $30\sqrt{-\partial_\tau \mu}$, which we show studying perturbations of linear operators. The denominator in (2) is strictly negative.

Numerical simulations of a spacetime with metric (1) show that the evolution becomes approximately Kasner at each value of θ. The signature for this is that both μ and also $v = \sqrt{(\partial_\tau P)^2 + e^{2P}(\partial_\tau Q)^2}$ are approximately constant in time, and $\partial_\tau \lambda \approx -v^2$. Both numerical simulations and qualitative analysis indicate that there are two types of transition. The signature of a transition is that v is not constant in time and $\partial_\tau \lambda$ is not approximated by $-v^2$. For each type of transition a rule for v is obtained by considering an exact “transition solution” which approximates the evolution. One type of transition is driven by the spatial curvature. It has been studied in various classes of spacetimes, including the Gowdy spacetimes, and gives two consecutive Kasner spacetimes in the BKL sequence. In the Gowdy spacetimes the oscillations eventually cease. If $\partial_\tau \mu \neq 0$ then a second type of transition occurs, in which $\partial_\tau \mu$ grows in magnitude and decays again, and throughout which the evolution remains Kasner, but the Kasner directions rotate. This rotation is geometrical. A given Kasner direction may be tangent to the symmetry orbit before the transition, but not tangent after. In the spatially homogeneous setting this type of transition was noticed using Hamiltonian methods[7] (“centrifugal potential”) and has since been noticed using dynamical systems methods.[8]

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